Encl (1)

(2)

AD-A197 498

(When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Advances in the p and h-p Versions of the Finite Element Method. A Survey		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) I. Babuska		8. CONTRACT OR GRANT NUMBER(s) ONR N00014-85-K-0169 NSF DMS-84-10324
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE 1988
		13. NUMBER OF PAGES 12
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper gives the survey of the advances in the theory and practice of the p and h-p versions of the finite element method. It gives the extensive list of references related to recent results of this new approach.

ADVANCES IN THE p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD. A SURVEY

I. Babuska
Institute for Physical Science and Technology
University of Maryland, College Park, MD 20742

ABSTRACT. The paper gives the survey of the advances in the theory and practice of the p and h-p versions of the finite element method. It gives the extensive list of references related to recent results of this new approach.

1. INTRODUCTION

The finite element method has become the main tool in computational mechanics. The MAKABASE ~~[42] [43]~~ contains at present the information on approximately 1400 finite/boundary element programs, about 20,000 references on finite element and 2,000 boundary element technology. To date there are more than two hundred monographs and proceedings, [44]. Recently a new direction in the finite element theory and practice appeared, the p and h-p versions which utilizes high degree elements. About 3-4 dozens references (out of 22,000) and only few programs are available. The aim of the paper is to briefly survey the state of the art about the p and h-p versions and present the basic references.

2. THE MODEL PROBLEM AND ITS PROPERTIES

We restrict ourself to the most simple but a characteristic model problem for elliptic partial differential equations. Let $\Omega \in \mathbf{R}^2$, $((x_1,x_2) = x)$ be a simply connected, bounded domain with the boundary $\partial\Omega = \Gamma = \bigcup_{i=1}^{M} \bar{\Gamma}_i$; Γ_i are analytic simple arcs called edges

$$\bar{\Gamma}_i = \{(\varphi_i(\xi),\psi_i(\xi)) | \xi \in \bar{I} = [-1,1]\}$$

where $\varphi_i(\xi)$, $\psi_i(\xi)$ are analytic functions on $\bar{I}$ and $|\varphi_i'(\xi)|^2 + |\psi_i'(\xi)|^2 \geq \alpha_i > 0$. Let A_i, $i = 1,\dots,M$ be vertices of Ω and $\Gamma_i = A_iA_{i+1}$ i.e. the edge Γ_i is linking the vertices A_i and A_{i+1}. By ω_i, $i = 1,\dots,M$ we denote the internal angles of Ω at A_i.

Let us be interested in the model problem

$$-\Delta u = f \text{ on } \Omega 1 \tag{2.1}$$

$$u = \varphi \text{ on } \Gamma^0 = \bigcup_{j \in Q} \bar{\Gamma}_j = \emptyset \tag{2.2}$$

AMS (M)S Subject Classification 65 N 30.

Partially supported by the Office of Naval Research under Contract N00014-85-K0169 and National Science Foundation under Grant DMS-84-10324.

$$\frac{\partial u}{\partial n} = \psi \text{ on } \Gamma^1 = \Gamma - \Gamma^0.$$

Here Q is a subset of $(1,\ldots,M) = \mathcal{M}$. The performance of the finite element method strongly depends on the properties of the (exact) solution of the solved problem especially on its smoothness.

In structural mechanics the problems are typically characterized by piecewise analytic input data. In our case the boundary is piecewise analytic and we will also assume that f is analytic on $\bar{\Omega}$ and φ and ψ are analytic on $\bar{\Gamma}_i$. The question arises, how to best characterize the solution of the problem (2.1) (2.2) for such input data. Is it possible to do in the terms of countably normed spaces.

Let $A_i = (x_{1,i}, x_{2,i})$, $r_i^2 = (x_1 - x_{1,i})^2 + (x_2 - x_{2,i})^2$, $\beta = \{\beta_i\}$, $0 < \beta_i < 1$, $i = 1,\ldots,M$. Define $\Phi_\beta(x) = \prod_{i=1}^{M} r_i^{\beta_i}(x)$ and for any integer k let $\Phi_{\beta\pm k}(x) = \prod_{i=1}^{M} r_i^{\beta_i + k}(x)$. Let finally $B^2_\beta(\Omega) = (u \in H^1(\Omega) | \|\Phi_{\beta+k-2} D^\alpha u\| \le Cd^k k!$ $k = 2,\ldots, |\alpha| = k$, C and d independent of k). We denoted $D^\alpha u = \partial^{\alpha_1+\alpha_2} u / \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$, $\alpha_1 + \alpha_2 = |\alpha|$. Function $u \in B^2_\beta(\Omega)$ is analytic on $\bar{\Omega} - \bigcup_{i=1}^{M} A_i$ and has a specific behavior in the neighborhood of the vertices A_i, $i = 1,\ldots,M$. Essential is that the solution of the problems of structural mechanics belongs to $B^2_\beta(\Omega)$. In [8] [11] we have proven

Theorem 2.1. Let f be analytic on $\bar{\Omega}$, φ_j be analytic on $\bar{\Gamma}_j$, $j \in Q$ and continuous on $\bar{\Gamma}^0$, ψ be analytic on $\bar{\Gamma}_j$, $j \in \mathcal{M} - Q$. Let u be the solution of (2.1) (2.2). Then $u \in B^2_\beta(\Omega)$ with $\beta_i > \bar{\beta}_i$ where $\bar{\beta}_i$ depends on ω_i and the type of boundary conditions on Γ_{i-1} Γ_i. $\square$

We can now ask the question how to completely characterize the class of the functions φ and ψ and f which guarantees that the solution of (2.1) (2.2) belongs to $B^2_\beta(\Omega)$. This has been done in [9]. Let us mention this characterization in a special (simplified) setting.

Theorem 2.2. Let Ω be a polygon, $u \in B^2_\beta(\Omega)$ and g_i be the trace of u on Γ_i. Then

a) if $1/2 < \beta_i, \beta_{i+1} < 1$, then $g_i \in B^1_{\hat{\beta}_i}(\Gamma_i)$, $\hat{\beta}_{i,j} \in (\beta_{i+j-1} - 1/2, 1/2)$, $j = 1,2$.

b) if $0 < \beta_i, \beta_{i+1} < 1/2$, then $g_i \in B^2_{\hat{\beta}_i}(\Gamma_i)$, $\hat{\beta}_{i,j} \in (\beta_{i+j-1} + 1/2, 1)$. $\square$

Here we denoted analogously as before for $s = 1,2$

$$B^s_\beta(\Gamma_i) = \{g \in H^{s-1}(\Gamma_i) | \|\prod_{j=1}^{2} |x - A_{i+j-1}|^{\hat{\beta}_{i,j} + \ell - s} g^{[\ell]}\|_{L_2(\Gamma_i)} \le Cd^\ell \ell!\}.$$

In [9] we also have shown the extension of the traces g belonging to the spaces mentioned in Theorem 2.2 to the function from $B^2_\beta(\Omega)$. By this characterization and Theorem 2.1 it is easy to verify that the solution of (2.1) (2.2) belongs to $\mathcal{B}^2_\beta(\Omega)$.

In [14] we have proven that the eigenfunctions belong to the space $\mathcal{B}^2_\beta(\Omega)$ too.

Although we have discussed only the special problem (2.1) (2.2), Theorem 2.1 is valid under much more general assumption and also for system of equations [10]. We also have characterized the countably normed spaces which characterize the solution of (2.1) (2.2) in 3 dimensions when the input data are piecewise analytic. This includes polyhedron domains.

3. THE FINITE ELEMENT METHOD

There are various different forms of the finite element method. We will consider here only the basic class of finite element methods (for our model problem).

Let $\mathcal{T} = \{\tau_i\}$ be a partition of Ω into (in general curvilinear) triangles or quadrilaterals called elements τ_i. In the case when $\mathcal{T}$ is a triangulation, we are making the standard assumptions. For the general case we refer to [11] [12], [36]. We will formulate here the assertions in the case of triangulation only, although they hold in general. Let $H(p,\mathcal{T}) = \{u \in H^1(\Omega) | u|_{\tau_i}, \tau_i \in \mathcal{T}$, is a polynomial of degree $p\}$ be the finite element space. If τ_i is a rectangle, then polynomials are of degree p in both variables. If the elements are curvilinear, then $u|_{\tau_i}$ are the standard "pull-back" polynomials.

We have assumed that the degree of polynomials are the same over all elements. The theory is also developed for general case when the degree p can be different on different elements.

Let $H_0(p,\mathcal{T}) = H(p,\mathcal{T}) \cap H^1_0(\Omega)$ where $H^1_0(\Omega) = \{u \in H^1(\Omega) | u = 0$ on $\Gamma^0\}$ and $\tilde{H}_0(p,\mathcal{T})$ be the restriction of $H(p,\mathcal{T})$ on Γ^0. We will assume that a projection operator $P_0(p,\mathcal{T})$ of function φ into $\tilde{H}_0(p,\mathcal{T})$ be given and we denote $\varphi_{p,\mathcal{T}} = P_0(p,\mathcal{T})\varphi$.

The finite element method consists now in finding $u_{FE} = u(p,\mathcal{T}) \in H(p,\mathcal{T})$ so that

1) $$u_{FE}(p,\mathcal{T}) = \varphi_{p,\mathcal{T}} \quad \text{on} \quad \Gamma^0$$

2) $$\iint_\Omega \left(\frac{\partial u_{FE}}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u_{FE}}{\partial x_2}\frac{\partial v}{\partial x_2}\right) dx_1 dx_1 = \int_{\Gamma^1} \psi v \, ds + \iint_\Omega fv \, dx_1 dx_2$$

holds for any $v \in H_0(p,\mathcal{T})$.

We will be interested here in the accuracy of the finite element solution measured in the energy norm. Define $e = u - u_{FE}$ and let

$$\|e(p,\mathcal{T})\|^2_E = \iint_\Omega \left(\left(\frac{\partial e}{\partial x_1}\right)^2 + \left(\frac{\partial e}{\partial x_2}\right)^2\right) dx_1 dx_2$$

be the error measured in the energy norm.

Two kind of operators $P_0(p,\mathcal{T})$ will be considered. Let $\gamma \subset \Gamma^0$ be the side of the triangle τ with endpoints A,B and assume that $\gamma = [-1,1]$. A $= -1$, B $= 1$. Then $\varphi_{p,\mathcal{T}}|_\gamma = \ell(x) + w(x)$ where $\ell(x)$ is linear function on γ such that $\varphi_{p,\tau}|_\gamma(\pm1) = \varphi(\pm1)$ and a) in the case of H^1-projection: $P_0^1(p,T): \varphi'(x) = \sum_{k=0}^{\infty} a_k \ell_k'(x)$, $w'(x) = \sum_0^{p-1} a_k \ell_k'(x)$ where ℓ_k are the Legendre polynomials; b) in the case of $H^{1/2}$-projection $P_0^{1/2}(p,T)$: $\varphi(x) = \sum_{k=0}^{\infty} a_n T_k(x)$, $w(x) = \sum_{k=0}^{p} a_k T_k(x)$, where $T_k(x)$ are the Tchebyschev polynomials. Other projections are analyzed in [12].

As we have seen there is a large freedom in the selection of $H(p,T)$ namely the degree p and the partition T and in the selection of the operators $P_0(p,T)$. We expect that $\|e\|_E \to 0$ if $\dim H_0(p,T) \to \infty$.

It is convenient to distinguish in this context three versions of the finite element method.

a) The h-version. Here a sequence (family) $H(p,T_i)$ is considered when p is fixed (usually $p = 1,2$) and the mesh T_i is successively refined so that the size h of the elements of T_i goes to zero.

b) The p-version. Here the mesh T is kept fixed and $p \to \infty$ uniformly on selectively.

c) The h-p version. In this version the mesh is simultaneously refined and the degree p increased uniformly or selectively.

The h-version of the finite element method is the standard one. The p-version is a recent development. The first theoretical paper about the p-version [28] and the h-p version [7] appeared in 1981 and various results were obtained since then.

There are many codes, research and commercial utilizing the h-version. The only commercial code using p and h-p versions is the code PROBE which was developed by NOETIC Technologies, St. Louis, MO [49] [60]. PROBE solves two dimensional problems of linear elasticity, stationary heat problems and thermoelasticity problems in two dimensions and elasticity in three dimensions. Three dimensional research code STRIPE was also developed by Swedish Aeronautical Research Institute, see e.g. [1] [2] [3]. These codes have in addition various features as adaptive approaches, various a-posteriori error estimation, etc.

4. THE BASIC PROPERTIES OF THE p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD

After 1980 the p and h-p versions of the finite element method was studied in detail from various point of view. We will mention here some essential illustrative results.

In one dimensional setting the versions were analyzed in detail in [34]. Here, among other, the optimal meshes and p-distribution has been established with upper and lower bounds of the errors for the three basic finite element versions.

In two dimensional setting the following theorem is characteristic for the performance of the h-p version. (For details see [9], [11], [12], [15], [35], [36], [37].)

Theorem 4.1. Let the solution u of the problems (2.1), (2.2) belongs to the set $B_\beta^2(\Omega)$. Then there is a sequence of meshes T_i and the degrees p_i such that

$$\|e\|_E < C e^{-\alpha \sqrt[3]{N_i}}, \quad \alpha > 0 \tag{4.1}$$

where $N_i = \dim H_0(p_i, \mathcal{T}_i)$ is the number of degrees-of-freedom for the h-p version. □

In one dimension the rate is $Ce^{-\alpha \sqrt[2]{N}}$. It has been proven in [34] that the optimal mesh is a geometric one with the factor $(\sqrt{2} - 1)^2 \simeq .17$. The experience shows that the geometric mesh with the factor $\simeq .15$ is also optimal in two dimension. Theorem 4.1 together with Theorem 2.1 shows that practically in any problem of structural mechanics the exponential rate of convergence can be achieved.

For the p-version the following theorem is another typical one (for more, see [23], [24], [25], [26]).

Theorem 4.2. Let us consider the problem (2.1) (2.2) and let $\bar{\beta} = \max(\beta_i)$ given in Theorem 2.1. Then for the p-version we have

$$\|e\|_E \le CN^{-(1-\bar{\beta})} \tag{4.2}$$

while for the h-version with uniform mesh

$$\|e\|_E \ge CN^{-\left(\frac{1-\bar{\beta}}{2}\right)}. \tag{4.3}$$ □

In Theorems 4.1 and 4.2 either the projection P_0^1 or $P_0^{1/2}$ could be used provided φ, is sufficiently smooth. The difference between these two operators occurs for the p-version when the boundary condition φ is unsmooth, e.g. $\varphi \in H^\delta(\Gamma)$, $1/2 < \delta < 3/4$. In this case the $H^{1/2}$ projection has to be used. For the h-p version there is no difference in the asymptotic rate but some difference occurs in the constant of the estimates. For the analysis of the influence of the operator P_0 on the accuracy we refer to [12] [15].

So far we have assumed that the domain Ω is bounded. Nevertheless the exponential rate of convergence (4.1) holds also for the problem on $\Omega^C = \mathbb{R}^2 - \Omega$ when f has bounded support. Here the infinite elements and properly selected shape functions have to be used. For more, see [17].

The h, p and h-p versions have different aspects with respect to the pollution problem. In presence of a singular behavior of the solution (e.g. in the neighborhood of the entrant corner of the domains) the L_∞ error is very large in the element consisting the corner. This effect disappears in elements which are separated away from the singularity by few elements. This effect is essential for a proper mesh design in practical computation. For details we refer to [16].

So far we have dealt with the problem (2.1) and (2.2) of second order. For the analyses of the finite element solution for the problems of order $2k$ we refer in the case of the h-p version to [35] and the p-version to [59]. For the basic analysis of the p-version in 3 dimensions, we refer to [31] and [32]. The eigenvalue problem is, as is well known, directly related to the "source" problem we addressed earlier. See e.g. [19] and [20]. The eigenfunctions belong to $\mathcal{B}^2_\beta(\Omega)$ and

$$|\lambda-\lambda_{FE}| \leq C e^{-2\alpha\sqrt[3]{N}}$$

$$\|u_{FE}(\lambda_h) - u(\lambda)\|_E \leq C e^{-\alpha\sqrt[3]{N}} .$$

For more details, see e.g. [14].

5. THE IMPLEMENTATION

There are some essential features of implementation of the p and h-p versions. See e.g. [60]. The elements are of a hierarchical type which leads to augmentation (bordering) of the local stiffness matrices when p is increased. This also allows to change very flexibly the degree of the shape functions from one element to the other one. The shape elements are (in two dimensions) of nodal type, side type and internal type and are based (in PROBE) on the integrals of the Legendre polynomials. This is important for numerical stability aspects.

The solvers can be direct or iterative. In [6] we have discussed the complexity of these solvers in the framework of parallel computation in the dependence on the architecture of the computer. In [48] we have proven various aspects of the preconditional conjugate gradient method. It has been shown that is is essential to eliminate first all element-internal unknowns — which can be done completely in parallel — and then for properly designed conjugate gradient method the number of iterations depend only on $\lg^2 p$ (p degrees of elements).

6. THE PROBLEM OF THE MESH DESIGN

One of the most laborious part of the finite element analysis, especially in three dimensions, is the mesh generation also when sophisticated mesh generators are used. The use of large elements (possibly of high degrees) which are describing only the geometry, greatly simplifies the user's work, also if possibly on expense of the computer time. (It is necessary to realize that the relation between manpower cost and computation cost is going steadily up.) The option of a change of the degrees of elements increases significantly the flexibility of the program and gives the user effective tool for the quality control. The p and h-p version programs give such options. It is advantageous therefore to create directly or indirectly the proper mesh and to achieve then the desired accuracy by an appropriate choice of the degrees, which can be made in an adaptive mode. The goal is to achieve the same combination of the degrees and mesh refinement which would be obtained for the given accuracy by the h-p version directly. To achieve this goal two avenues could be followed, the expert system and the adaptive approach. <u>The expert system</u>, see e.g. [21], [51], advises the user how to design the mesh and element degrees for the requested accuracy and provides the user with a mesh generator.

The adaptive approach (see e.g. [1], [2], [3], [34] Part 3) which is possible to see as an "automatic" expert system makes various decisions for the users. Both approaches have some common parts but the concepts are significantly different. We refer also to [46], [52], [56], [65], [66], [67] for various additional aspects.

7. THE ROBUSTNESS

An effective method has to perform uniformly well for a broad class of input data. The elasticity problems can be in practice nearly singular as, for example, in the case of nearly incompressible material, various plate and shell theories, in the case of thin domains, etc. The h-version suffers in these cases by the "locking" problem which has to be overcome by various special approaches as reduced integration, etc. Problems of these types are avoided when the p and h-p versions are used. The convergence rate in the energy norm then (in contrast to the h-versions) is uniform with respect to the Poisson ratio when higher degrees of elements are used. See e.g. [56], [39], [73]. For computation of the pressure for $\nu \approx 1/2$, see [69].

8. THE QUALITY CONTROL OF THE SOLUTION

It is essential to have a possibility of a quantitative assessment of the quality of computed data. For the survey of today's general ideas and results in this direction, we refer to [45]. In the case of the p and h-p versions there is relatively easy way for the quality assessment of any data of the interest by changing the degrees and by an extrapolation procedure. This approach is very effective because it indicates reliably the errors of any computed data of interest, the energy norm, value of stresses, stress intensity factor, etc. See also [1], [11],[33], [36], [38], [49], [58], [60], [61], [64], [68].

9. p AND h-p VERSION IN OTHER PROBLEMS

So far we mentioned only the use in the context of elliptic problems [57]. [77] addresses the use of p-version for the boundary element method, [74], [75], [76] in the context of the Kirchhoff plate analysis. [53], [71], show that the p and h-p versions are very effective tools for computation of thin structures as plates and shells and [54] for optimal design. [13] analyzes the p-version for parabolic equations when high order polynomials are used in space and time and [41] is using the p-version for solving elliptic stochastic equations.

10. THE COMPUTATIONAL AND ENGINEERING EXPERIENCE

Because of the developed commercial code PROBE and research code STRIPE an extensive experience is already available in the research and industrial use. For the industrial experience we refer e.g. to [29], [49] where numerical results are presented. For the numerical data we refer e.g. to [1], [2], [3], [4], [11], [12], [13], [14], [15], [16], [17], [21], [22], [36], [38], [41], [48], [51], [53], [54] [61], [62],[63], [66], [67], [68], [71], [72]. The experience shows that the p and h-p versions of the finite element method has many practical advantages for the engineering computations for linear elliptic problems.

11. RELATION TO SOME OTHER METHODS

The ideas of the p and h-p versions are related for example to various methods used in fluid dynamics as the spectral method and its variation. We refer e.g. to [46] and references given there. A commercial fluid dynamics code NEKTON which is based on spectral method and developed by NEKTONICS INC, Badford, MA is closely related to the idea of h-p version of the finite element method. See also [30].

REFERENCES

[1] Andersson, B., Error estimation and adaptivity in FE-analysis of structures-Research program, Rep. FFAH-85-03. The Aeronautical Research Institute of Sweden, Structures Dept., Box 11021, 16111 Bromma, Sweden.

[2] Andersson, B., Falk, U., Jarlas, R., Self-adaptive FE-analysis of solid structures, Rep. FFA TM 1986-27. The Aeronautical Research Institute of Sweden, Structures Dept., Box 11021, 16111 Bromma, Sweden.

[3] Andersson, B., Falk, U., Finite element analysis of three-dimensional structures using adaptive p-extensions, Rep. FFA TN 1986-57. The Aeronautical Research Institute of Sweden, Structures Dept. Box 11021, 16111 Bromma, Sweden.

[4] Babuška, I., The p and h-p versions of the finite element methods. The state of the Art. Proc. of the Workshop on Finite Element Method, Nasa Langley, ed. R. Voigt, Springer 1988, 199-239.

[5] Babuška, I., Are High Degree Elements Preferable? Some Aspects of the h and h-p versions of the Finite Element Method in Numerical Techniques for Engineering Analysis and Design. Proceedings of the International Conference on Numerical Methods in Engineering: Theory and Applications. Numeta 87, Swansee, 6-10 July 1987. G. N. Pande & J. Middleton eds. Martinus Nijhoff Publishers, Dordrecht/Boston/ Lancaster, 1987, paper S1.

[6] Babuška, I., Elman, H. C. Some Aspects of Parallel Implementation of the Finite Element Method on Message Passing Architectures.

[7] Babuska, I., Dorr, M. R., Error Estimates for the Combined h and p-Version of the Finite Element Method, Numer. Math. 37 (1981), 257-277.

[8] Babuška, I., Guo. B., Regularity of the solution of elliptic problems with piecewise analytic data, Part I: Boundary value problem for linear elliptic equations of second order. SIAM J. Math. Anal. 19 (1988), 172-203.

[9] Babuška, I., Guo, B. Q. Regularity of the solution of elliptic problems with piecewise analytic data, Part II: The trace spaces and applications to the boundary value problems with nonhomogeneous boundary conditions.

[10] Babuška, I., Guo, B., Regularity of the solution of elliptic problems with piecewise analytic data. Part III. Boundary value problem for system of equations of second order. To appear.

[11] Babuška, I., Guo, B., The h-p version of the Finite Element Method for Domains with Curved Boundaries. To appear in SIAM J. Numer. Anal., 1988 (Tech Note BN-1057, 1986, Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742).

[12] Babuška, I., Guo, B., The h-p version of the finite element method for problems with nonhomogeneous essential boundary conditions.

[13] Babuška, I., Janik, T. The p-version of the finite element method for parabolic equations.

[14] Babuška, I., Guo, B. Q., Osborn, J. E. Regularity and numerical solution of eigenvalue problems with piecewise analytic data. Tech. Note. BN-1073, 1988, Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742.

[15] Babuška, I., Guo, B. Q., Suri, M., Implementation of nonhomogeneous Dirichlet boundary conditions in the p-version of finite element method.

[16] Babuška, I., Oh, Hae-Soo, Pollution problems for the p-version and the h-p version of finite element method. Communications in Applied Numerical Methods 3 (1987), 553-561.

[17] Babuška, I., Oh, Hae-Soo, The p and h-p versions of the finite element method for elliptic problems on infinite domains.

[18] Babuška, I., Gui, W., Szabó, B. A., Performance of the h-p and h, p and h-p versions of the finite element method in Research in Structures and Dynamics, (R. J. Hayduk, A. K. Noor, eds) NASA Conference Publication 2325 (1984), 73-94.

[19] Babuška, I., Osborn, J., Estimates foı the errors in eigenvalue and eigenvector approximation by Galerkin methods, with particular attention to the case of multiple eigenvalues. SIAM J. Num. Anal. 24 (1987), 1249-1276.

[20] Babuška, I., Osborn, J., Eigenvalue Problems. To appear in Handbook of Numerical Analysis, ed. P. G. Ciarlet, J. L. Lions, North-Holland, Amsterdam [Tech Not BN-1066, 1987, Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742).

[21] Babuška, I., Rank, E. An expert-system-like feedback approach in the h-p version of the finite element method. Finite Elements in Analysis and design 3 (1987), 127-147.

[22] Babuška, I., Scapolla, T., Computational aspects of the h, p and h-p versions of the finite element method. Advances in Computer Methods in Partial Differential Equations-VI, R. Vichnevetsky and R. S. Stepleman eds. Int'l Association for Mathematics and Computer Simulation, 1987, 233-240.

[23] Babuška, I., Suri, M., The optimal convergence rate of the p-version of the finite element method. SIAM J. Numer. Anal. 24 (1987), 750-776.

[24] Babuška, I., Suri, M., The h-p version of the finite element method with quasi uniform meshes. Math. Modeling Numer. Anal. (RAIRO) 21 (1987), 199-238.

[25] Babuška, I., Suri, M., The treatment of nonhomogeneous Dirichlet boundary condition by the p-version of the finite element method, To appear in Math. Comp. 1988.

[26] Babuška, I., Suri, M., The p-version of the finite method for constrained boundary conditions (Tech. Note BN-1064, 1987, Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742).

[27] Babuška, I., Szabó, B. A., Rates of convergence of the finite element method, Internat. J. Numer. Methods Engrg. 18 (1982), 323-341.

[28] Babuška, I., Szabó B. A., Katz, I. N., The p-version of the finite element method. SIAM J. Numer. Anal. 18 (1981), 512-545.

[29] Barnhart, M. A., Eisenmann, J. R., Analysis of a Stiffened Plate Detail Using p-Version and h-Version Finite Element Techniques. Presented at the First World Congress on Computational Mechanics, Sept. 22-26, 1986. The University of Texas at Austin.

[30] Delves, L. M., Phillips, C., A fast implementation of the global element method, J. Inst. Math. Appl. 25 (1980), 177-197.

[31] Dorr, M. R., The approximation theory for the p-version of the finite element method, SIAM J. Numer. Anal. 21 (1984), 1181-1207.

[32] Dorr, M. R., The approximation of solutions of elliptic boundary value problems via the p-version of the finite element method, SIAM J. Numer. Anal. 23 (1986), 58-77.

[33] Dunavant, D. A., Szabó B. A., A-posteriori error indicators for the p-version of the finite element method, Internat. J. Numer. Methods Engrg. 10 (1983), 1851-1870.

[34] Gui, W., Babuška, I., The h, p and h-p versions of the finite element method in 1 dimension. Part 1: The error analysis of the p-version. Part 2: The error analysis of the h- and h-p versions. Part 3: The adaptive h-p version, Numerische Mathematik, 48 (1986), 577-612, 613-657, 619-683.

[35] Guo, B., The h-p version of the finite element method for elliptic equations of order 2m. To appear in Numerische Mathematik.

[36] Guo, G., Babuška, I., The h-p version of the finite element method. Part 1: The basic approximation results. Part 2: General results and applications, Computational Mechanics 1 (1986), 21-41, 203-226.

[37] Guo, B., Babuska, I., The theory and practice of the h-p version of finite element method. Advances in computer methods in partial differential equations VI, R. Vichnevetsky and R. S. Stepleman, eds. Int'l Association for Mathematics and Computer Simulation (1987), 241-247.

[38] Izadpanah, K., Computation of the Stress Components of the Finite Element Method, Ph.D. Dissertation, Washington University, St. Louis, 1984.

[39] Jensen, S., Vogelius, M., Divergence stability in connection with the p-version of the finite element method. Rep. MD 87-47-SJ/MV, TR 87-47, University of Maryland, College Park, MD 20742.

[40] Katz, I. N., Wang. E. W., The p-version of the finite element method for problems requiring C^1-continuity, SIAM J. Numer. Anal. 22 (1985), 1082-1106.

[41] Larsen, S., Numerical Analysis of Elliptic Partial Differential Equations with Stochastic Input Data. Ph.D. Thesis, University of Maryland, 1986.

[42] Mackerle J., MAKABASE, An information system on structural mechanics software and applications. Adv. Eng. Software 8 (1986), 81-87.

[43] Mackerle J., MAKABASE, An On-line Information Retrieval System of Structural Mechanics Computers and Structures 24 (1986), 977-983.

[44] Noor, A. K., Books and monographs on finite element technology, Finite Elements in Analysis and Design, 1 (1985), 101-111.

[45] Noor, A. K., Babuska, I., Quality assessment and control of finite element solutions, Finite Elements in Analysis and Design 3 (1987), 1-26.

[46] Patera, A. T., Advances and Future Directions of Research on Spectral Methods. Computational Mechanics: Advances and Trends. A. K. Noor ed. AMD-Vol 75, ASME 1987, 411-427.

[47] Peano, A. G., Walker, J. W., Modeling of Solid Continua by the p-Version of the Finite Element Method. Proc. 3rd World Congress on Finite Element Methods, Beverly Hills, California, 1981.

[48] Pitkäranta, J., Babuška, I., The conjugate gradient method for the p-version of the finite element method.

[49] PROBE - Sample Problems. Series of reports, Noetic Technologies, St. Louis, MO 63117.

[50] Rank, E., Babuška, I., An expert system for the optimal mesh design in the h-p version of the finite element method. Int. J. Num. Engrg. 24 (1987), 2087-2106.

[51] Rank, E., An Adaptive hp-Version in the Finite Element Method, in Numerical Techniques for Engineering Analysis and Design. Proceedings of the International Conference on Numerical Methods in Engineering: Theory and Applications. Numeta 87, Swansee, 6-10 July 1987. G. N. Pande & J. Middleton eds. Martinus Nijhoff Publishers, Dordrecht/-Boston/Lancaster, 1987, paper S14.

[52] Rank, E., Adaptive Boundary Element Methods, Rep. Nr 87/3, 1987, Fachgebiet Elektronisches Rechnen im Konstruktiven Ingenieurbau Technische Hochshule, München, Arcisstrasse 21, D-8000 - München 2.

[53] Sahramann, G. J., Hierarchic Plate and Shell Models Based on p-Extension. Ph.D. dissertation, Washington University, St. Louis, MO, May 1988.

[54] Schiermeier, J. E., Szabó, B. A., Interactive Design based on the p-version of the finite element method. Finite Elements in Analysis and Design 3 (1987), 93-107.

[55] Scott, L. R., Vogelius, M., Norm estimates for a maximal inverse of the divergence operator in spaces of piecewise polynomials, Math. Modelling Num. Anal. (RAIRO) 19 (1985), 111-143.

[56] Scott, L. R., Vogelius, M., Conforming Finite Element Methods for Incompressible and Nearly Incompressible Continua. Lectures in Applied Mathematics 22 (1985), 221-244.

[57] Stephan, E. P., Suri, M., On the convergence of the p-version of the boundary element Galerkin method. To appear in Math. Comp.

[58] Suri, M. The p-Version of the Finite Element Method for Elliptic Problems. Advances in Computer Methods in Partial Differential Equations VI, R. Vichnevetsky, R. S. Stepleman, eds. Int'l Association for Mathematical and Computer Simulation, 1987, 85-91.

[59] Suri, M. The p-Version of the Finite Element Method of elliptic equations of order 2ℓ. To appear.

[60] Szabó, B. A., Probe: Theoretical Manual, 1985, NOETIC Tech, 7980 Clayton Road, Suite 205, St. Louis, MO 63117.

[61] Szabó, B. A., Mesh design for the p-version of the finite element method, Computer Methods in Applied Mathematics and Engineering 55 (1986), 181-197.

[62] Szabó, B. A., On the Errors of Idealization and Discretization in Finite Element Analysis Advances in Advances in Computer Methods in Partial Differential Equations-VI, R. Vichnevetsky and R. S. Stepleman, eds. Int'l Association for Mathematics and Computer Simulation, 1987, 70-74.

[63] Szabó, B. A., Implementation of a finite element software system with h- and p-extension capabilities, Finite Elements in Analysis and Design 2 (1986), 177-194.

[64] Szabó, B. A., Estimation and Control of Error Based on P-convergence in Accuracy Estimates and Adaptive Refinements in Finite Element computations (I. Babuška, J. Gago, E. R. de A. Oliveira, D. V. Zienkiwicz, eds.) John Wiley & Sons, 61-70.

[65] Szabó, B. A., On Geometric Idealization in Finite Element Computations, Tech. Note WU/CCM-87/4, Center for Computational Mechanics, Washington University, St. Louis, MO 63130. Presented at the Workshop on Special Topics in Computational Mechanics, Dallas, TX, April 13-14, 1987.

[66] Szabó, B. A., Computation of stress field parameters in areas of steep stress gradients. Comm. in Appl. Numer. Methods 2 (1986), 133-137.

[67] Szabó, B. A., Design of Finite Element Software for Reliability and Robustness, Proc Internat. Conference on Reliability and Robustness of Engineering Software, Como, Italy, Sep. 23-25, 1987.

[68] Szabó, B. A. and Babuška, I., Computation of the Amplitude of Stress Singular Terms for Cracks and Reentrant Corners. Fracture Mechanics, XIX Symposium ASTM, STP 1969, T.A. Cruse ed., Am. Soc. Test and Mat., Philadelphia, 1987, 101-126.

[69] Szabó, B. A., Babuška, I., Stress computation for nearly incompressible materials, Part 1: Under restriction on smoothness.

[70] Szabó, B. A. and Peano, A. G., Hierarchic Finite Elements, Rep WU/CCM-83/1 Center for Computational Mechanics, Washington University, St. Louis, MO 63130.

[71] Szabó, B. A., Sahrmann, G. J., Hierarchical Plate and Shell Models Based on p-Extension. To appear in Internat. Journ. Num. Meth. Engrg.

[72] Vasilopoulos, D., Treatment of Geometric Singularities with the p-Version of Finite Element Method. Ph.D. Dissertation, Washington University, St. Louis, MO, 1984.

[73] Vogelius, M., An analysis of the p-version of the finite element method for nearly incompressible materials-uniformly valid, optimal estimates, Numer. Math. 41 (1983), 39-53.

[74] Wang, D. W. Katz, I. N., Szabó B. A., Implementation of C^1 triangular elements based on the p-version of the finite element method, Comp. & Structures 10 (1984), 381-392.

[75] Wang, D. W., Katz, I. N., Szabó, B. A., h and p-versions finite element analysis of a rhombic plate, Internat. J. Numer. Methods Engrg. 20 (1984), 1399-1405.

[76] Wang, w. D., Katz, I. N., Szabó, B. A., Implementation of a C^1 Triangular Element Based on the p-version of the Finite Element Method. SIAM J. Num. Anal. 22 (1985), 1082-1106.

[77] Wendland, W., Splines versus trigonometric polynomials. h-versus p-version in 1D boundary integral methods. Preprint Nr. 925, 1985, TH Darmstadt Fachbereich Mathematik.